AN ACCURATELY SOLVABLE PROBLEM OF THE MUTUAL EFFECT OF INCLUSIONS IN THE THEORY OF HETEROGENEOUS MEDIA

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1. Introduction. One of the main problems of the theory of disperse mixtures in the study of physical fields consists of correct accounting of the mutual effect of inclusions. In real heterogeneous media this problem is compounded by the difference in shapes and size of inclusions, the diversity of structures formed by them, and the difference in material properties of the constituent components and the carrying phase. In view of the diversity of governing factors, field calculations in inhomogeneous media have practically not been performed, and to simplify the calculations it is necessary to adopt a number of assumptions. If, for example, the characteristic sizes of inclusions are small and do not differ much from each other, it is usually assumed that all elements of the disperse phase are identical and usually have the shape of spheres, ellipsoids, or cylinders. As is well known, the field in thus separately selected bodies is homogeneous when they are placed in a homogeneous external field. This property is used in investigating heterogeneous media with a low inclusion concentration. If, moreover, the characteristics of the inclusion material vary within narrow limits, they can be assumed to be of one type. Thus, within the first approximation the disperse mixture is treated as two components or, in the general case, a two-phase system consisting of a matrix and one-kind inclusions.

For three-component media with two-size inclusions the field problems become multiparametric, and their solution is extremely complicated. The concept of generalized parameters of such heterogeneous systems and explaining the nature of processes occurring in them can be obtained by analyzing several model problems. The present study is devoted to solving one of the similar problems, resulting from the study of inhomogeneous films or thin coatings (these results are also valid for bulk composites, identically hardened by oriented filaments, if the fields are calculated in the transverse plane).

The problem of determining the electric field in a conducting medium, containing two arbitrarily located circular inclusions and electrical resistances of different radii is solved. In the presence of an external magnetic field the Hall effect may appear in all three components. It is noted that this problem was investigated algebraically in [1, 2] by using R-functions in the case of intersecting single-type inclusions.

In the present study the required two-dimensional electric field is effectively calculated by methods of the theory of complex variables, allowing one to obtain a closed analytic solution without any restrictions on the problem conditions. Though calculations are carried out for the electric field, due to known analogies one can investigate thermal, diffusion, magnetic field, and other physical field equations quite similarly.

2. Statement and Solution of the Problem. In an unconfined medium with specific electric resistance ρ_0, let there be two circular inclusions with radii r_1 and r_2 and resistances ρ_1 and ρ_2, respectively (Fig. 1a). The centers of the circular inclusions are located at the points $z = 0$ and $z = h > r_1 + r_2$ $(z = x + iy)$. It is necessary to find the current distribution inside the inclusions and in their neighborhoods for an arbitrary direction of the external current J, which is assumed given.

In each of the three regions the vectors of current density $\mathbf{j}$ and electric field intensity $\mathbf{E}$ are potential and solenoidal; therefore, in the complex variable z plane one can introduce the holomorphic functions $j(z) = j_x(x, y) - ij_y(x, y)$ and $E(z) = E_x(x, y) - iE_y(x, y)$, the relation between them given by Ohm's law $E(z) = \rho j(z)$, coinciding in shape, due to the reality of the parameter ρ, with the vector form $\mathbf{E} = \rho\mathbf{j}$. The field equations $\mathrm{rot}\,\mathbf{E} = 0$ and $\mathrm{div}\,\mathbf{j} = 0$ lead in this case to the Cauchy–Riemann conditions for the functions $j(z)$ and $E(z)$.

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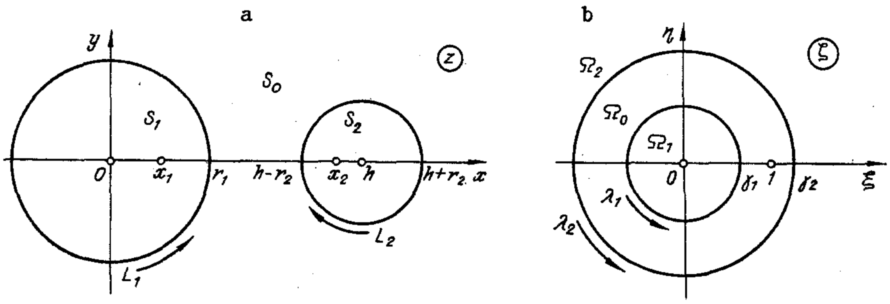

Fig. 1

The ohmic contact conditions are satisfied at the boundary of various type media, continuity of the normal components of the current vector $\mathbf{j}$ and of the tangential components of the field vector $\mathbf{E}$.

Thus, in the plane z = x + iy it is necessary to determine the piecewise-holomorphic function

$$j(z)=\begin{cases} j_0(z), & z\in S_0=\{z:|z|>r_1,\ |z-h|>r_2\},\\ j_1(z), & z\in S_1=\{z:|z|<r_1\},\\ j_2(z), & z\in S_2=\{z:|z-h|<r_2\},\end{cases} \tag{2.1}$$

satisfying the boundary conditions

$$\begin{aligned} \operatorname{Re}\{n(t)j_0(t)\} &= \operatorname{Re}\{n(t)j_k(t)\},\\ \operatorname{Im}\{n(t)\rho_0 j_0(t)\} &= \operatorname{Im}\{n(t)\rho_k j_k(t)\},\ t\in L_k\ (k=1,2). \end{aligned} \tag{2.2}$$

Here n(t) is the unit normal to the contour: $L = L_1 \cup L_2$:

$$n(t)=\frac{t}{r_1},\quad t=r_1 e^{i\theta},\quad t\in L_1,$$

$$n(t)=\frac{t-h}{r_2},\quad t=h+r_2 e^{i\theta},\quad t\in L_2,\quad 0\leqslant\theta<2\pi.$$

On the basis of the last equalities and the identities $\bar{t}=r_1^2/t,\ t\in L_1,\ \overline{t-h}=r_2^2/(t-h)$, $t\in L_2$, obtained by an inverse transformation, the boundary conditions (2.2) can be rewritten in expanded form

$$\begin{aligned} & j_0(t)+\left(\frac{r_1}{t}\right)^2\overline{j_0(t)}=j_1(t)+\left(\frac{r_1}{t}\right)^2\overline{j_1(t)},\\ & \rho_0 j_0(t)-\rho_0\left(\frac{r_1}{t}\right)^2\overline{j_0(t)}=\rho_1 j_1(t)-\rho_1\left(\frac{r_1}{t}\right)^2\overline{j_1(t)},\quad t\in L_1,\\ & j_0(t)+\left(\frac{r_2}{t-h}\right)^2\overline{j_0(t)}=j_2(t)+\left(\frac{r_2}{t-h}\right)^2\overline{j_2(t)},\\ & \rho_0 j_0(t)-\rho_0\left(\frac{r_2}{t-h}\right)^2\overline{j_0(t)}=\rho_2 j_2(t)-\rho_2\left(\frac{r_2}{t-h}\right)^2\overline{j_2(t)},\quad t\in L_2. \end{aligned} \tag{2.3}$$

The bar over a variable or a function denotes complex conjugation.

The external electric current J flowing in the system is given by its value at infinity

$$j(\infty)=j_0(\infty)=J=J_x-iJ_y. \tag{2.4}$$

The boundary conditions are somewhat simplified if from each pair of equalities (2.3) one eliminates the function $\overline{j_0(t)}$:

$$\begin{aligned} 2\rho_0 j_0(t) &= (\rho_0+\rho_1)j_1(t)+(\rho_0-\rho_1)\left(\frac{r_1}{t}\right)^2\overline{j_1(t)},\quad t\in L_1,\\ 2\rho_0 j_0(t) &= (\rho_0+\rho_2)j_2(t)+(\rho_0-\rho_2)\left(\frac{r_2}{t-h}\right)^2\overline{j_2(t)},\quad t\in L_2. \end{aligned} \tag{2.5}$$

It is easily shown that conditions (2.3) and (2.5) are equivalent. The boundary-value problem (2.5) refers to a special case of the Markushevich problem, or, as it is also called, the general coupling problem [3, p. 222].

For subsequent analysis of the results it is convenient to introduce the relative values of specific resistances of the inhomogeneous inclusions

$$\Delta_{0k} = (\rho_0 - \rho_k)/(\rho_0 + \rho_k), \; -1 \leqslant \Delta_{0k} \leqslant 1 \;\; (k = 1, 2).$$

The boundary relations (2.5) acquire then the form

$$\begin{aligned}(1 + \Delta_{01})j_0(t) &= j_1(t) + \Delta_{01}(r_1/t)^2\overline{j_1(t)}, \; t \in L_1, \\ (1 + \Delta_{02})j_0(t) &= j_2(t) + \Delta_{02}(r_2/(t-h))^2\overline{j_2(t)}, \; t \in L_2.\end{aligned} \tag{2.6}$$

Below we provide a solution of the boundary-value problem (2.6), which, unlike the general case of the Markushevich problem, can be constructed explicitly. It is first necessary to map conformally the exterior of the circular inclusions S_0 on a concentric ring. This mapping is realized, as is well-known, by the fractionally linear function

$$\zeta = T^{-1}(z) = (z - x_1)/(z - x_2), \; \zeta = \xi + i\eta, \tag{2.7}$$

where x_1 and x_2 are symmetric points with respect to both surroundings L_1 and L_2:

$$x_1x_2 = r_1^2, \; (h - x_1)(h - x_2) = r_2^2. \tag{2.8}$$

According to equalities (2.8), the points x_1 and x_2 are located on the real Ox axis in the circular regions S_1 and S_2, respectively, and have coordinates

$$x_{1,2} = (1/2h)\{h^2 + r_1^2 - r_2^2 \mp \sqrt{(h^2 + r_1^2 - r_2^2)^2 - 4h^2r_1^2}\}. \tag{2.9}$$

The inverse mapping is given by the function

$$z = T(\zeta) = (x_2\zeta - x_1)/(\zeta - 1). \tag{2.10}$$

The point position for mapping (2.7) is shown in Fig. 1a, b. In this case the radii of the surroundings λ_1 and λ_2, into which the neighborhoods L_1 and L_2 transform, are determined by the equations:

$$\gamma_1 = \sqrt{x_1/x_2}, \; \gamma_2 = \sqrt{(h - x_1)/(h - x_2)} \;\; (\gamma_1 < 1, \; \gamma_2 > 1), \tag{2.11}$$

and the point at infinity $z = \infty$ transform to the point $\zeta = 1$, i.e., it is located in the ring $\gamma_1 < |\zeta| < \gamma_2$.

In the transformed region the problem reduces to searching a piecewise-holomorphic function

$$f(\zeta) = j(T(\zeta)) = \begin{cases} f_0(\zeta), & \zeta \in \Omega_0 = \{\zeta : \gamma_1 < \zeta < \gamma_2\}, \\ f_1(\zeta), & \zeta \in \Omega_1 = \{\zeta : |\zeta| < \gamma_1\}, \\ f_2(\zeta), & \zeta \in \Omega_2 = \{\zeta : |\zeta| > \gamma_2\}. \end{cases} \tag{2.12}$$

Based on Eqs. (2.4) and (2.6), the following boundary-value problem is obtained for the function $f(\zeta)$

$$\begin{aligned}(1 + \Delta_{01}) f_0(\tau) &= f_1(\tau) + \Delta_{01}\left(\frac{r_1}{T(\tau)}\right)^2\overline{f_1(\tau)}, \;\; \tau \in \lambda_1, \\ (1 + \Delta_{02}) f_0(\tau) &= f_2(\tau) + \Delta_{02}\left(\frac{r_2}{T(\tau) - h}\right)^2 f_2(\tau), \; \tau \in \lambda_2, \;\; f(1) = f_0(1) = J.\end{aligned} \tag{2.13}$$

From the possibility of expanding the function $f_0(\zeta)$ in a Laurent series it follows that it admits the representation

$$f_0(\zeta) = f_0^+(\zeta) + f_0^-(\zeta), \tag{2.14}$$

where the functions $f_0^+(\zeta)$ and $f_0^-(\zeta)$ are holomorphic in the regions $\{\zeta : |\zeta| < \gamma_2\}$ and $\{\zeta : |\zeta| > \gamma_1\}$, respectively. Without loss of generality one can put

$$f_0^+(1) = 0, \; f_0^-(1) = J. \tag{2.15}$$

With account of the partitioning (2.14), the boundary conditions (2.13) are transformed to the form

$$\begin{aligned}(1 + \Delta_{01}) f_0^+(\tau) - f_1(\tau) &= \Delta_{01}\left(\frac{r_1}{T(\tau)}\right)^2\overline{f_1(\tau)} - (1 + \Delta_{01}) f_0^-(\tau), \;\; \tau \in \lambda_1, \\ (1 + \Delta_{02}) f_0^-(\tau) - f_2(\tau) &= \Delta_{02}\left(\frac{r_2}{T(\tau) - h}\right)^2\overline{f_2(\tau)} - (1 + \Delta_{02}) f_0^+(\tau), \;\; \tau \in \lambda_2.\end{aligned} \tag{2.16}$$

From relations (2.16) and the theorem of analytic continuity it follows that the functions

$$
\begin{aligned}
&\Phi(\zeta)=\begin{cases}(1+\Delta_{01})f_0^+(\zeta)-f_1(\zeta), & |\zeta|<\gamma_1,\\ \Delta_{01}\left(\dfrac{r_1}{T(\zeta)}\right)^2\overline{f_1\left(\dfrac{\gamma_1^2}{\bar\zeta}\right)}-(1+\Delta_{01})f_0^-(\zeta), & |\zeta|>\gamma_1,\end{cases}\\
&\Psi(\zeta)=\begin{cases}(1+\Delta_{02})f_0^-(\zeta)-f_2(\zeta), & |\zeta|>\gamma_2,\\ \Delta_{02}\left(\dfrac{r_2}{T(\zeta)-h}\right)^2\overline{f_2\left(\dfrac{\gamma_2^2}{\bar\zeta}\right)}-(1+\Delta_{02})f_0^+(\zeta), & |\zeta|<\gamma_2\end{cases}
\end{aligned}
\tag{2.17}
$$

are holomorphic in the whole ζ-plane, since the functions $\overline{f_1(\gamma_1^2/\bar\zeta)}$ and

$$r_1/T(\zeta)=\gamma_1(\zeta-1)/(\zeta-\gamma_1^2) \tag{2.18}$$

are holomorphic at $|\zeta|>\gamma_1$, while $\overline{f_2(\gamma_2^2/\bar\zeta)}$ and

$$r_2/(T(\zeta)-h)=-\gamma_2(\zeta-1)/(\zeta-\gamma_2^2) \tag{2.19}$$

are holomorphic for $|\zeta|<\gamma_2$.

By the Liouville theorem

$$\Phi(\zeta)\equiv C_1,\ \Psi(\zeta)\equiv C_2 \tag{2.20}$$

(C_1 and C_2 are complex constants). Putting in Eqs. (2.17) and (2.20) $\zeta = 1$, on the basis of equalities (2.15) and (2.18), (2.19) one easily obtains

$$C_1=-(1+\Delta_{01})J,\ C_2=0. \tag{2.21}$$

Thus, Eqs. (2.17) acquire the form

$$
\begin{aligned}
&(1+\Delta_{01})f_0^+(\zeta)-f_1(\zeta)=-(1+\Delta_{01})J, && |\zeta|<\gamma_1,\\
&\Delta_{01}\left(\frac{r_1}{T(\zeta)}\right)^2\overline{f_1\left(\frac{\gamma_1^2}{\bar\zeta}\right)}-(1+\Delta_{01})f^-(\zeta)=-(1+\Delta_{01})J, && |\zeta|>\gamma_1,\\
&(1+\Delta_{02})f_0^-(\zeta)-f_2(\zeta)=0, && |\zeta|>\gamma_2,\\
&\Delta_{02}\left(\frac{r_2}{T(\zeta)-h}\right)^2\overline{f_2\left(\frac{\gamma_2^2}{\bar\zeta}\right)}-(1+\Delta_{02})f_0^+(\zeta)=0, && |\zeta|<\gamma_2.
\end{aligned}
\tag{2.22}
$$

From relations (2.14) and (2.22) follows the validity of the following representation of the piecewise-holomorphic function (2.12):

$$
f(\zeta)=\begin{cases}(1+\Delta_{01})\,[J+f_0^+(\zeta)], & |\zeta|<\gamma_1,\\ f_0^+(\zeta)+f_0^-(\zeta), & \gamma_1<|\zeta|<\gamma_2,\\ (1+\Delta_{02})\,f_0^-(\zeta), & |\zeta|>\gamma_2.\end{cases}
\tag{2.23}
$$

The functions $f^{\pm}(\zeta)$ are found from expressions (2.22), where to the second and third of them one applies an inversion transformation with respect to the neighborhoods γ_1 and γ_2, respectively. Expressions (2.22) acquire then the form

$$
\begin{aligned}
&(1+\Delta_{01})f_0^+(\zeta)-f_1(\zeta)=-(1+\Delta_{01})J, && |\zeta|<\gamma_1,\\
&\Delta_{01}\left(\frac{T(\zeta)}{r_1}\right)^2 f_1(\zeta)-(1+\Delta_{01})\overline{f_0^-\left(\frac{\gamma_1^2}{\bar\zeta}\right)}=-(1+\Delta_{01})\bar J, && |\zeta|<\gamma_1,\\
&(1+\Delta_{02})\overline{f_0^-\left(\frac{\gamma_2^2}{\bar\zeta}\right)}-\overline{f_2\left(\frac{\gamma_2^2}{\bar\zeta}\right)}=0, && |\zeta|<\gamma_2,\\
&\Delta_{02}\left(\frac{r_2}{T(\zeta)-h}\right)^2\overline{f_2\left(\frac{\gamma_2^2}{\bar\zeta}\right)}-(1+\Delta_{02})f_0^+(\zeta)=0, && |\zeta|<\gamma_2.
\end{aligned}
\tag{2.24}
$$

From the first pair of expressions (2.24) one can eliminate the function $f_1(\zeta)$, and from the second, $\overline{f_2(\gamma_2^2/\bar\zeta)}$. As a result we have the relations

$$
\begin{aligned}
&f_0^+(\zeta)+\frac{1}{\Delta_{01}}\left(\frac{r_1}{T(\zeta)}\right)^2\left\{\bar J-\overline{f_0^-\left(\frac{\gamma_1^2}{\bar\zeta}\right)}\right\}=-J, && |\zeta|<\gamma_1,\\
&\overline{f_0^-\left(\frac{\gamma_2^2}{\bar\zeta}\right)}-\frac{1}{\Delta_{02}}\left(\frac{T(\zeta)-h}{r_2}\right)^2 f_0^+(\zeta)=0, && |\zeta|<\gamma_2.
\end{aligned}
\tag{2.25}
$$

Hence, following elimination of the function $f_0^+(\zeta)$ and some easy transformations taking into account Eqs. (2.18) and (2.19), the following functional equation is obtained

$$f_0^-(\zeta) = J + \Delta_{01}\bar{J}\left(\gamma_1 \frac{\zeta-1}{\zeta-\gamma_1^2}\right)^2 + \Delta_{01}\Delta_{02}\left(\Gamma\frac{\zeta-1}{\zeta-\Gamma^2}\right)^2 f_0^-(\Gamma^{-2}\zeta), \quad |\zeta| > \gamma_1 \tag{2.26}$$

where

$$\Gamma = \frac{\gamma_1}{\gamma_2} = \sqrt{\frac{x_1(h-x_2)}{x_2(h-x_1)}} < 1. \tag{2.27}$$

Before turning to solve Eq. (2.26), it can be proved that it, implying also problem (2.13), cannot have more than a single solution. Indeed, the opposite would imply that Eq. (2.26) has a nontrivial solution for J = 0. The latter is impossible, since for J = 0 and $\zeta = \infty$ we find from Eq. (2.26) $f_0^-(\zeta) = \Delta_{01}\Delta_{02}\Gamma^2 f_0^-(\infty)$, whence due to the inequality $|\Delta_{01}\Delta_{02}\Gamma^2| < 1$ and the boundedness of $f_0^-(\zeta)$ it follows that $f_0^-(\zeta) = 0$. From this it follows, in turn, the holomorphicity near infinity of the function $\varphi_1(\zeta) = \zeta f_0^-(\zeta)$, satisfying, due to Eq. (2.26) (J = 0), the relation

$$\varphi_1(\zeta) = \Delta_{01}\Delta_{02}\Gamma^2\left(\Gamma\frac{\zeta-1}{\zeta-\Gamma^2}\right)^2 \varphi_1(\Gamma^{-2}\zeta),$$

comparing left and right hand sides of which for $\xi = \infty$ gives $\varphi_1(\infty) = 0$. Similarly, one subsequently shows the holomorphicity near infinity of the functions $\varphi_k(\zeta) = \zeta^k f_0^-(\zeta)$ and the equality $\varphi_k(\infty) = 0$ (k = 2, 3,...), which, obviously, is equivalent to the statement made above.

By the method of mathematical induction it is shown that as a result of eliminating the functions $f_0^-(\Gamma^{-2k}\zeta)$, $k = \overline{1, n-1}$, from the set of n equations obtained from (2.26) by replacing ζ by $\Gamma^{-2k}\zeta$, $k = \overline{0, n-1}$, Eq. (2.26) leads to

$$f_0^-(\zeta) = \sum_{k=0}^{n-1}\left\{(\Delta_{01}\Delta_{02}\Gamma^2)^k\left[J\left(\frac{\zeta-1}{\zeta-\Gamma^{2k}}\right)^2 + \bar{J}\Delta_{01}\gamma_1^2\left(\frac{\zeta-1}{\zeta-\gamma_1^2\Gamma^{2k}}\right)^2\right]\right\} + (\Delta_{01}\Delta_{02}\Gamma^2)^n\left(\frac{\zeta-1}{\zeta-\Gamma^{2n}}\right)^2 f_0^-(\Gamma^{-2n}\zeta), \ |\zeta| > \gamma_1. \tag{2.28}$$

Transforming in Eq. (2.28) to the limit $n \to \infty$, for the function $f_0^-(\zeta)$ we obtain the expression

$$f_0^-(\zeta) = \sum_{k=0}^{\infty}\left\{(\Delta_{01}\Delta_{02}\Gamma^2)^k\left[J\left(\frac{\zeta-1}{\zeta-\Gamma^{2k}}\right)^2 + \bar{J}\Delta_{01}\gamma_1^2\left(\frac{\zeta-1}{\zeta-\gamma_1^2\Gamma^{2k}}\right)^2\right]\right\}, \ |\zeta| > \gamma_1. \tag{2.29}$$

The series (2.29) converges absolutely and uniformly due to the inequalities

$$\max_{|\zeta|\geqslant\gamma_1}\left|\frac{\zeta-1}{\zeta-\Gamma^{2k}}\right| < \max_{|\zeta|\geqslant\gamma_1}\left|\frac{\zeta-1}{\zeta-\gamma_1^2\Gamma^{2k}}\right| = \frac{\gamma_1+1}{\gamma_1+\gamma_1^2\Gamma^{2k}} < \frac{\gamma_1+1}{\gamma_1}.$$

Knowing $f_0^-(\zeta)$, one easily determines the function $f_0^+(\zeta)$ by turning to any of the two relations (2.25):

$$f_0^+(\zeta) = (\zeta-1)^2\sum_{k=1}^{\infty}\left\{(\Delta_{01}\Delta_{02})^k\left[J\left(\frac{\Gamma^{-k}}{\zeta-\Gamma^{-2k}}\right)^2 + \frac{\bar{J}}{\Delta_{01}}\left(\frac{\gamma_1\Gamma^{-k}}{\zeta-\gamma_1^2\Gamma^{-2k}}\right)^2\right]\right\}, \ |\zeta| < \gamma_2. \tag{2.30}$$

On the basis of Eqs. (2.29), (2.30) the piecewise-holomorphic function $f(\zeta)$ is now reconstructed according to the representation (2.23). The inverse mapping to the physical plane $z = x + iy$ makes it possible to find the required current distribution inside the inclusions and in their neighborhoods. Finally, the calculations lead to the equations

$$j(z) = J + (x_2-x_1)^2\left\{\sum_{k=1}^{\infty}\left\{(\Delta_{01}\Delta_{02})^k\left[J\left(\frac{Q(\Gamma^{-k})}{z-T(\Gamma^{-2k})}\right)^2\right] + \frac{\bar{J}}{\Delta_{01}}\left(\frac{Q(\gamma_1\Gamma^{-k})}{z-T(\gamma_1^2\Gamma^{-2k})}\right)^2\right]\right\} +$$
$$+ \sum_{k=1}^{\infty}\left\{(\Delta_{01}\Delta_{02})^k\left[J\left(\frac{Q(\Gamma^{k})}{z-T(\Gamma^{2k})}\right)^2 + \frac{\bar{J}}{\Delta_{02}}\left(\frac{Q(\gamma_2\Gamma^{k})}{z-T(\gamma_2^2\Gamma^{2k})}\right)^2\right]\right\}\right\}, \quad |z| > r_1, \ |z-h| > r_2, \tag{2.31}$$
$$(1+\Delta_{01})\left\{J + (x_2-x_1)^2\sum_{k=1}^{\infty}\left\{(\Delta_{01}\Delta_{02})^k\left[J\left(\frac{Q(\Gamma^{-k})}{z-T(\Gamma^{-2k})}\right)^2 + \frac{\bar{J}}{\Delta_{01}}\left(\frac{Q(\gamma_1\Gamma^{-k})}{z-T(\gamma_1^2\Gamma^{-2k})}\right)^2\right]\right\}\right\}, \ |z| < r_1,$$
$$(1+\Delta_{02})\left\{J + (x_2-x_1)^2\sum_{k=1}^{\infty}\left\{(\Delta_{01}\Delta_{02})^k\left[J\left(\frac{Q(\Gamma^{k})}{z-T(\Gamma^{2k})}\right)^2 + \frac{\bar{J}}{\Delta_{02}}\left(\frac{Q(\gamma_2\Gamma^{k})}{z-T(\gamma_2^2\Gamma^{2k})}\right)^2\right]\right\}\right\}, \ |z-h| < r_2.$$

Here $Q(w) = w/(1 - w^2)$, and the remaining notations are given by expressions (2.9)-(2.11) and (2.27).

Equations (2.31), despite their awkward appearance, really have a simple structure and a clear physical interpretation. The current distribution outside the exclusion region S_0 contains a constant component J, whose value is acquired at infinity, and dipole components at the points $T(\cdot)$. One pair of dipoles with coordinates $T(\Gamma^{2k})$ and $T(\gamma_2\Gamma^{2k})$ is located on the segment $[0, x_1]$ of the real axis. With increasing power k the absolute values of the dipole moments decrease, and the dipoles are displaced from the center of the circular inclusion 0 toward the point x_1, which is a vanishing point for the dipoles. Another pair of dipoles has coordinates $T(\Gamma^{2k})$ and $T(\gamma_2\Gamma^{2k})$, i.e., is located on the segment $[x_2, h]$ of the real axis. In this case, with increasing power k the dipoles are compressed toward the point x_2, while the absolute values of their moments decrease in this case. Thus, the mutual effect of inhomogeneous inclusions in a continuous medium is displayed in the dipole-dipole interaction.

The current distribution near the inclusions has a power dependence on the inclusion radii r_1 and r_2, the distance between them is h, and on the relative specific inhomogeneity resistances Δ_{01} and Δ_{02}. Most differences in the current distributions are observed in directions of the external current J along the Ox and Oy axes, i.e., for subsequent alternation of the inclusions and their parallel arrangement with respect to the current J.

Inside the inclusions the expressions for the current contain constant components $(1 + \Delta_{0k})J$ $(k = 1, 2)$, corresponding accurately to the current distribution in a separately selected circular inclusion in the absence of the other. These current components do not exceed in magnitude twice the current value J at infinity, and coincide with them in direction. The action of the inclusions on each other is manifested in that there exist additional current components, represented by dipoles which are located near the given inclusion. The dipoles are, accurately within the factor $(1 + \Delta_{0k})$, the same as in the expression for the current in the exterior of the inclusion region. The current inside adjacent inclusions can exceed 2J for certain relations between the system parameters.

It is necessary to stress that the absolute values of the dipole moments decrease sharply with increasing power k, and that the main contribution to the current value is provided by dipoles located in the centers of the circles. As shown by direct calculations, this property is manifested in a wide variation of all parameters appearing in the current expression. Therefore, in many practical calculations it is sufficient to confine oneself to the first approximate equation of (2.31) only

$$j(z)=\begin{cases} J+\bar{J}\left\{\dfrac{\Delta_{01}r_1^2}{z^2}+\dfrac{\Delta_{02}r_2^2}{(z-h)^2}\right\}, & z\in S_0,\\ (1+\Delta_{01})\left\{J+\bar{J}\dfrac{\Delta_{02}r_2^2}{(z-h)^2}\right\}, & |z|<r_1,\\ (1+\Delta_{02})\left\{J+\bar{J}\dfrac{\Delta_{01}r_1^2}{z^2}\right\}, & |z-h|<r_2. \end{cases} \tag{2.32}$$

These equations are convenient to use, and have a simple physical interpretation. If in expression (2.31) one transforms to one of the three limits: $r_2 \to 0$, $h \to \infty$, or $\Delta_{02} \to 0$, they provide the solution for the case of a single inclusion [4]. Such a smoother transition is displayed in Eqs. (2.32).

To represent the features of the current distribution inside inclusions and in their neighborhoods we show in Fig. 2 plots of the relative magnitude of the current density vector on the Ox axis, constructed by Eq. (2.31) under the following conditions. The external current vector J is along the Ox axis $(J = J_x)$, i.e., the inclusions alternate successively with respect to the vector J. The current density vector has then only one component along the Ox axis $j(x) = j_x(x)$, and its relative magnitude is $j^* = j/J = j_x/J_x$. In the computer calculations it was assumed that both inclusions have identical radii r, and that the distance between their centers is $h = 3r$.

Curves 1-3 correspond to equal specific inclusion resistances. Curve 1 corresponds to the case in which the specific resistances of the inclusions are identical and less in magnitude than the resistance of the fundamental component ($\rho_1 = \rho_2 < \rho_0$, $\Delta_{01} = \Delta_{02} = 0.5$), 2 was constructed under the condition that the inclusions have different specific resistances: ρ_1, $\rho_2 < \rho_0$, $\Delta_{01} = 0.5$, $\Delta_{02} = -0.5$, while 3 also refers to the case of identical conducting inclu-

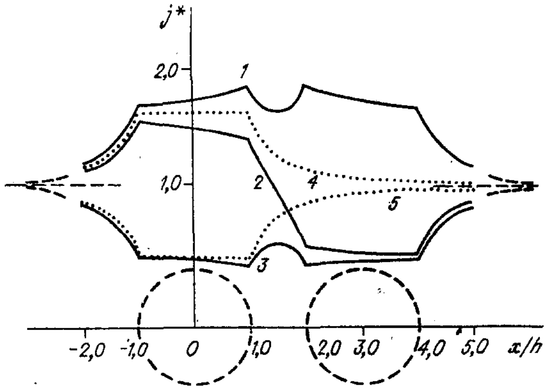

Fig. 2

sions, but under the condition that their specific resistance exceed the resistance of the medium in which they are submerged ($\rho_1 = \rho_2 > \rho_0$, $\Delta_{01} = \Delta_{02} = -0.5$).

As could be expected, inclusions with high conductivity concentrate the current inside them; the current flows away from low-conducting inclusions. The nature of the mutual inclusion effect on the current distribution can be concluded by comparing curves 1-3 with curves 4 and 5, constructed for the case in which there exists only one inclusion of radius r with center at the origin of coordinates, having relative specific resistances $\Delta_{01} = 0.5$ and -0.5 (curves 4, 5). Compared with an isolated inclusion, the current concentration in mutually interacting inclusions is enhanced or reduced, depending largely on the specific inclusion resistances. Qualitatively, this is also verified by direct computer calculations, whose results are given in [1, 2].

It is interesting to note that the calculations for constructing curves 1-3 by the exact (2.31) and approximate (2.32) equations differ by approximately 1%. Thus, the first series terms in expressions (2.31) provide the main contribution in the calculations.

3. Account of the Hall Effect. The solution method discussed is also fully applicable to studying problems with anisotropic conductivity, when the material anisotropy is caused by Hall effects in a magnetic field $\mathbf{B} = \{0, 0, B\}$, $B = \text{const}$. In this case the specific electric resistance has a tensor representation, and the Ohm law acquires the form $E(z) = \hat{\rho} j(z)$ [$\hat{\rho} = \rho(1 + i\beta)$, β is the Hall parameter]. With account of the complex specific resistance the boundary conditions (2.2) have a different shape:

$$\begin{aligned} &\operatorname{Re}\{n(t)j_0(t)\} = \operatorname{Re}\{n(t)j_k(t)\}, \\ &\operatorname{Im}\{n(t)\bar{\bar{\rho}}_0 j_0(t)\} = \operatorname{Im}\{n(t)\bar{\bar{\rho}} j_k(t)\},\ t \in L_k\ (k = 1, 2). \end{aligned} \tag{3.1}$$

In what follows the behavior of the solution of problem (3.1) remains as previously, and the final result is written down by Eqs. (2.31) and (2.32), with the only difference that the scalar quantities Δ_{0k} must be replaced by the complex quantities $\Delta_{0k} \to \hat{\Delta}_{0k}$ [$\hat{\Delta}_{0k} = (\Delta_{0k} - iB_{0k})/(1 + iB_{0k})$ (k = 1, 2)]. Here B_{0k} are the reduced values of the Hall parameter: $B_{0k} = (\rho_0\beta_0 - \rho_k\beta_k)/(\rho_0 + \rho_k)$.

In a medium with a Hall effect the current distribution pattern is more complicated, and has important features. Firstly, if the following relation is satisfied $\rho_0\beta_0 = \rho_k\beta_k$ [due to the equation $\beta_k = (1/\rho)R_{Hk}B$ this is equivalent to the equality of Hall coefficients: $R_{H0} = R_{Hk}$], then $B_{0k} = 0$, $\hat{\Delta}_{0k} = \Delta_{0k}$, and the external magnetic field has no effect whatsoever on the current distribution. Secondly, which is particularly interesting, for unrestricted growth in the magnetic field and increase in reduced Hall parameters B_{0k} the current flows from the inclusions independently of their resistances, and in the limit $B_{0k} \to \infty$ it behaves as in the case of flow in a conducting medium with dielectric inclusions.

Thus, the system considered provides one of the few three-component examples of an inhomogeneous medium, for which one can obtain exact analytic results. To study this structure it has been established that the effect of inclusions on each other has the nature of dipole–dipole interaction. In the presence of a large number of inclusions the problem is substantially more complicated, and does not admit solution by relatively simple methods. To some extent the situation is similar to that encountered in many-body mechanics. The solution difficulties, increasing with the number of interacting inclusions, are partially overcome by considering several symmetric structures in systems with a regular disperse phase distribution [4].

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COOLING OF A MAGNETIZED PLASMA AT A BOUNDARY WITH AN EXPLODING METAL WALL

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Cooling of a magnetized plasma at the boundary with a cold wall, which is accompanied by reaction of magnetic and thermal processes, leads in a number of cases to anomalously high effective thermal conductivity and magnetic diffusion coefficients. With cooling of a hydrogen plasma at a boundary with an insulator or a dense multicharged plasma, the effective thermal conductivity appears to be of the order of Bohm thermal conductivity [1, 2].

With cooling of a plasma bounded by a rigid and ideally conducting wall, as was shown in [1], the increase in thermal conductivity compared with classical magnetized thermal conductivity is less marked and it is only possible for a plasma with $\beta \gg 1$ ($\beta = 16\pi N_0 T_0/H_0^2$ is the ratio of thermal pressure of the plasma to magnetic pressure; N_0, T_0, and H_0 are density of electrons, temperature, and magnetic field in the plasma at a distance from the boundary). A metal wall may be considered to be rigid and ideally-conducting in the case when it does not explode due to action of heat flow from the plasma, i.e., its thermal conductivity in the condensed phase appears to be sufficient in order to remove heat without evaporating. This condition is fulfilled with relatively high energy densities (for plasma with $T_0 = 1$ keV and $\beta = 1$, with $H_0 \lesssim 0.2$ MG). With higher energy densities presence of an explosive heat flow for the metal markedly changes the nature of cooling and it increases heat losses for the plasma. This case is considered in the present work. However, the magnetic fields are not considered to be very high ($H_0 > 10$ MG) since with $H_0 < 10$ MG when there is explosion of a skin layer by Joule heat and the metal loses conductivity, the problem is reduced to that considered previously [1, 2] of plasma cooling at a boundary with an insulator.

Let all of the values depend on coordinate X perpendicular to the metal surface, and time t, magnetic field H, and electric field E are perpendicular to each other and axis X, and characteristic times are large compared with gas dynamic times, so that total pressure both in the hydrogen plasma and in metal vapor have time to equalize:

$$p + H^2/8\pi = P_0 \equiv 2N_0T_0 + H_0^2/8\pi \tag{0.1}$$

(p is thermal pressure). Equations for the magnetic and electric fields and the thermal balance for the plasma [3] are written in Lagrangian variables, and have the form

$$\frac{\partial E}{\partial X} = -\frac{1}{c}\left(\frac{dH}{dt} - \frac{H}{\rho}\frac{d\rho}{dt}\right), \quad \frac{\partial H}{\partial X} = -\frac{4\pi}{c}j, \quad E = j/\sigma - \frac{b_\Lambda}{e}\frac{\partial T}{\partial X},$$
$$\rho\frac{d\varepsilon}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = -\frac{\partial Q}{\partial X} + jE, \quad Q = -\chi\frac{\partial T}{\partial X} + \frac{b_\Lambda T}{e}j, \tag{0.2}$$

where ρ is density of the mass; ε is internal energy; σ, χ, b_Λ are transverse conduction, themal conductivity, and thermoelectric coefficients; Q is density of heat flow.

<u>1. Cooling of a Dense Plasma.</u> As shown in [1], existence of anomalously high effective thermal conductivity coefficients means that the problem for a hydrogen plasma is quasistationary: hydrogen plasma density in the boundary zone is large compared with density N_0, and

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